CDS: Numerical Methods - Assignment week 5

Solutions to the exercise have to be handed in via Brightspace in form of one or several executable Python scripts (*.py) which run without any errors. The deadline for the submission is **Thursday Mar. 12, 13:30**. Feel free to use the Science Gitlab repository to submit your solutions.

1 Ordinary Differential Equations – Initial Value Problems

In the following you will successively implement your own easy-to-extend ODE solver which is capable of using single-step as well as multi-step approaches and to handle first order ODE systems, i.e. we aim to solve:

$$\vec{y}(x)' = \vec{f}(x, \vec{y})$$

for $x = x_0, x_1, \dots, x_n$ with $x_i = ih$, step size h, and an initial condition of the form $\vec{y}(x = 0)$.

(a) Implement the Euler method by defining a general Python function `y = integrator(x, y0, f, phi)` which takes as input a one-dimensional x array of size $n + 1$, the initial condition y_0, the *callable* function $f(x, y)$, and the integration scheme $\Phi(x, y, h, f)$ and which returns the approximated function $\tilde{y}(x)$. The integration scheme `phi` is also supposed to be a *callable* function as returned from another Python function `phi = phi_euler(xi, yi, h, f)`.

(b) Debug your implementation by applying it to:

$$\vec{y}(x)' = y - x^2 + 1.0$$

with $y(x = 0) = 0.5$, which has the solution $y(x) = (x + 1)^2 - 0.5e^x$. To this end define the right-hand side of the ODE as a Python function `f = ODEF(x, y)` which returns here $f(x, y) = y - x^2 + 1.0$. Afterwards, you can hand over `f` to your integrator function.

(c) Extend the set of possible single-step integration schemes to the modified Euler (Collatz), Heun, and 4th-order Runge-Kutta approaches by implementing the functions `phi = phi_euler_modified(xi, yi, h, f)`, `phi = phi_heun(xi, yi, h, f)`, and `phi = phi_rk4(xi, yi, h, f)`.

(d) Add the possibility to also handle the following multi-step integration schemes:

$$\Phi_{AB3}(x, y, h, f, i) = \frac{1}{12}\left[23f(x_i, y_i) - 16f(x_{i-1}, y_{i-1}) + 5f(x_{i-2}, y_{i-2})\right]$$

and

$$\Phi_{AB4}(x, y, h, f, i) = \frac{1}{24}\left[55f(x_i, y_i) - 59f(x_{i-1}, y_{i-1}) + 37f(x_{i-2}, y_{i-2}) - 9f(x_{i-3}, y_{i-3})\right]$$

In these cases the initial condition `y0` must consist of several initial values corresponding to y_0, y_1, y_2 (and y_3) and the integration scheme $\Phi(x, y, h, f, i)$ must be called using the step number i. Use the Runga-Kutta method to calculate all of the initial values before you start the AB3 and AB4 integrations.

(e) Plot the absolute errors $\delta(x) = |\tilde{y}(x) - y(x)|$ with a y-log scale for all approaches with $0 \le x \le 2$ and a step size of $h = 0.02$ for the ODE from (b).

(f) Study the accuracies of all approaches as a function of the step size h. To this end use your implementations to solve

$$\vec{y}(x)' = y$$

with $y(x = 0) = 1.0$ for $0 \le x \le 2$ and plot $\delta(2) = |\tilde{y}(2) - y(2)|$ as a function of h for each integration scheme.

(g) Apply the Euler and the Runga-Kutta method to solve the pendulum problem given by:

$$\vec{y}(x)' = \vec{f}(x, \vec{y}) \quad \leftrightarrow \quad \begin{pmatrix} y_0(x)' \\ y_1(x)' \end{pmatrix} = \begin{pmatrix} y_1(x) \\ -\sin\left[y_0(x)\right] \end{pmatrix}$$

To this end all the quantities `y, y0, f, phi` must become vectors. Plot $y_0(x)$ for several oscillation periods. Does the Euler method behave physical?